

The Complete Weight Enumerator of Several Cyclic Codes

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Received: date / Accepted: date

Abstract Cyclic codes have attracted a lot of research interest for decades. In this paper, for an odd prime p , we propose a general strategy to compute the complete weight enumerator of cyclic codes via the value distribution of the corresponding exponential sums. As applications of this general strategy, we determine the complete weight enumerator of several p -ary cyclic codes and give some examples to illustrate our results.

Keywords Cyclic code · Gauss sum · Exponential sum · Weight enumerator · Complete weight enumerator

Mathematics Subject Classification 11T71-94B15

1 Introduction

Throughout this paper, let p be an odd prime. Denote by \mathbb{F}_p a finite field with p elements. An $[n, \kappa, \delta]$ linear code C over \mathbb{F}_p is a κ -dimensional subspace of \mathbb{F}_p^n with minimum distance δ . Moreover, the code is cyclic if every codeword $(c_0, c_1, \dots, c_{n-1}) \in C$ whenever $(c_{n-1}, c_0, \dots, c_{n-2}) \in C$. Any cyclic code C of length n over \mathbb{F}_p can be viewed as an ideal of $\mathbb{F}_p[x]/(x^n - 1)$. Therefore, $C = \langle g(x) \rangle$, where $g(x)$ is the monic polynomial of lowest degree and divides $x^n - 1$. Then $g(x)$ is called the generator polynomial and $h(x) = (x^n - 1)/g(x)$ is called the parity-check polynomial [21].

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The ordinary weight enumerator of C of length n is defined by

$$A_0 + A_1x + A_2x^2 + \cdots + A_nx^n,$$

where A_i is the number of codewords with Hamming weight i and $A_0 = 1$. The sequence $(A_0, A_1, A_2, \dots, A_n)$ is called the weight distribution of the code C .

The complete weight enumerator of a code C over \mathbb{F}_p enumerates the codewords according to the number times each element of the field appears in each codeword. Denote the field elements by $\mathbb{F}_p = \{w_0, w_1, \dots, w_{p-1}\}$, where $w_0 = 0$. Also let \mathbb{F}_p^* denote $\mathbb{F}_p \setminus \{0\}$. For a codeword $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in \mathbb{F}_p^n$, let $w[\mathbf{c}]$ be the complete weight enumerator of \mathbf{c} defined as

$$w[\mathbf{c}] = w_0^{k_0} w_1^{k_1} \cdots w_{p-1}^{k_{p-1}},$$

where k_j is the number of components of \mathbf{c} equal to w_j , $\sum_{j=0}^{p-1} k_j = n$. The complete weight enumerator of the code C is then

$$\text{CWE}(C) = \sum_{\mathbf{c} \in C} w[\mathbf{c}].$$

The weight distribution of a code has been extensively studied for a long time and we refer the reader to [8, 10, 12, 22, 23, 25] and references therein for an overview of the related researches. Note that the complete weight enumerator of a codeword implies its weight, which indicates that the weight distribution of the code can be obtained from its complete weight enumerator. The information of the complete weight enumerator of a linear code is of vital use in practical applications. For example, Blake and Kith pointed out that the complete weight enumerator of Reed-Solomon codes could be helpful in soft decision decoding [2, 15]. In [14], the study of the monomial and quadratic bent functions was related to the complete weight enumerators of linear codes. Ding *et al.* [7, 9] showed that the complete weight enumerator can be applied in the computation of the deception probabilities of certain authentication codes. In [3, 6, 11], the complete weight enumerators of some constant composition codes were shown to have only one term and some families of optimal constant composition codes were presented.

However, only a few works were focused on the determination of the complete weight enumerator of linear codes in the literature besides the above mentioned [2, 15, 3, 6, 11]. The complete weight enumerators of the generalized Kerdock code and related linear codes over Galois rings were determined by Kuzmin and Nechaev in [16, 17]. The authors obtained the complete weight enumerators of some cyclic codes by using exponential sums in [1, 18]. In this paper, we shall determine the complete weight enumerators of a class of cyclic codes related to some special quadratic forms.

Let m and l be two positive integers with $m > l$. For now on, we denote by α a primitive element of \mathbb{F}_{p^m} . Let $h_1(x)$ and $h_2(x)$ be the minimal polynomials of $\alpha^{-(p^l+1)}$ and α^{-2} over \mathbb{F}_p , respectively. Obviously, $h_1(x)$ and $h_2(x)$ are distinct and $\deg(h_2(x)) = m$. Moreover, it can be easily shown that $\deg(h_1(x)) = m/2$ if $m = 2l$ and m otherwise.

Let C_1 and C_2 be two cyclic codes over \mathbb{F}_p of length $p^m - 1$ with parity-check polynomials $h_1(x)$ and $h_1(x)h_2(x)$, respectively. Hence, for the dimensions of C_1 and C_2 , we have

$$\dim_{\mathbb{F}_p} C_1 = \begin{cases} \frac{1}{2}m, & \text{if } m = 2l, \\ m, & \text{otherwise,} \end{cases}$$

and

$$\dim_{\mathbb{F}_p} C_2 = \begin{cases} \frac{3}{2}m, & \text{if } m = 2l, \\ 2m, & \text{otherwise.} \end{cases}$$

From the well-known Delsartes Theorem [5], we have the trace representation of C_1 and C_2 described by

$$\begin{aligned} C_1 &= \{(\text{Tr}_1^m(ax^{p^l+1}))_{x \in \mathbb{F}_{p^m}^*} : a \in \mathbb{F}_{p^m}\}, \\ C_2 &= \{(\text{Tr}_1^m(ax^{p^l+1} + bx^2))_{x \in \mathbb{F}_{p^m}^*} : a, b \in \mathbb{F}_{p^m}\}. \end{aligned}$$

The weight distribution of C_1 is trivial and can be easily obtained since the value distributions of the corresponding exponential sums are already known (see [4, 13]). However, to the best of our knowledge, there are no information about its complete weight enumerator. The cyclic code C_2 was investigated in the literature. Luo and Feng [20] studied its weight distribution explicitly. Bae, Li and Yue [1] established its complete weight enumerator in the special case of $\gcd(m, l) = 1$. In this paper, we will explicitly present the complete weight enumerators of C_1 and C_2 in view of the relationship between $v_2(m)$ and $v_2(l)$ for arbitrary m and l with $m > l$, where $v_2(\cdot)$ is the 2-adic order function. Thus, we will extend the results in [1] to some extent.

The aim of this paper is to investigate the complete weight enumerators for cyclic codes by utilizing the theories of Gauss sums and exponential sums over finite fields. A general strategy is proposed and then applied to determine the complete weight enumerators for the codes C_1 and C_2 , respectively.

The remainder of this paper is organized as follows. In Section 2, we introduce some definitions and auxiliary results on quadratic forms, Gauss sums and exponential sums. Section 3 gives the main results of this paper, including a general strategy for cyclic codes and the explicitly complete weight enumerators for the codes C_1 and C_2 . Section 4 concludes this paper and makes some remarks on this topic.

2 Preliminaries

We follow the notations in Section 1. Let q be a power of p and t be a positive integer. By identifying the finite field \mathbb{F}_{q^t} with a t -dimensional vector space \mathbb{F}_q^t over \mathbb{F}_q , a function $f(x)$ from \mathbb{F}_{q^t} to \mathbb{F}_q can be regarded as a t -variable

polynomial over \mathbb{F}_q . The function $f(x)$ is called a quadratic form if it can be written as a homogeneous polynomial of degree two on \mathbb{F}_q^t as follows:

$$f(x_1, x_2, \dots, x_t) = \sum_{1 \leq i \leq j \leq t} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{F}_q.$$

Here we fix a basis of \mathbb{F}_q^t over \mathbb{F}_q and identify each $x \in \mathbb{F}_q^t$ with a vector $(x_1, x_2, \dots, x_t) \in \mathbb{F}_q^t$. The rank of the quadratic form $f(x)$, $\text{rank}(f)$, is defined as the codimension of the \mathbb{F}_q -vector space

$$W = \{x \in \mathbb{F}_q^t \mid f(x+z) - f(x) - f(z) = 0, \text{ for all } z \in \mathbb{F}_q^t\}.$$

Then $|W| = q^{t-\text{rank}(f)}$.

For a quadratic form $f(x)$ with t variables over \mathbb{F}_q , there exists a symmetric matrix A over \mathbb{F}_q such that $f(x) = XAX'$, where $X = (x_1, x_2, \dots, x_t) \in \mathbb{F}_q^t$ and X' denotes the transpose of X . It is known that there exists a nonsingular matrix B over \mathbb{F}_q such that BAB' is a diagonal matrix. Making a nonsingular linear substitution $X = YB$ with $Y = (y_1, y_2, \dots, y_t) \in \mathbb{F}_q^t$, we have

$$f(x) = Y(BAB')Y' = \sum_{i=1}^r a_i y_i^2, \quad a_i \in \mathbb{F}_q^*,$$

where r is the rank of $f(x)$. The determinant $\det(f)$ of $f(x)$ is defined to be the determinant of A , and $f(x)$ is said to be nondegenerate if $\det(f) \neq 0$.

The quadratic character over \mathbb{F}_{p^m} is defined by

$$\eta(x) = \begin{cases} 1, & \text{if } x \text{ is a square in } \mathbb{F}_{p^m}^*, \\ -1, & \text{if } x \text{ is a nonsquare in } \mathbb{F}_{p^m}^*, \\ 0, & \text{if } x = 0. \end{cases}$$

The canonical additive character of \mathbb{F}_{p^m} , denoted χ , is given by

$$\chi(x) = \zeta_p^{\text{Tr}_1^m(x)}$$

for all $x \in \mathbb{F}_{p^m}$, where $\zeta_p = e^{2\pi\sqrt{-1}/p}$ and Tr_1^m is a trace function from \mathbb{F}_{p^m} to \mathbb{F}_p defined by

$$\text{Tr}_1^m(x) = \sum_{i=0}^{m-1} x^{p^i}, \quad x \in \mathbb{F}_{p^m}.$$

To this end, we shall introduce the Gauss sum $G(\eta, \chi)$ over \mathbb{F}_{p^m} given by

$$G(\eta, \chi) = \sum_{x \in \mathbb{F}_{p^m}^*} \eta(x) \chi(x) = \sum_{x \in \mathbb{F}_{p^m}} \eta(x) \chi(x),$$

and the Gauss sum $G(\bar{\eta}, \bar{\chi})$ over \mathbb{F}_p given by

$$G(\bar{\eta}, \bar{\chi}) = \sum_{x \in \mathbb{F}_p^*} \bar{\eta}(x) \bar{\chi}(x) = \sum_{x \in \mathbb{F}_p} \bar{\eta}(x) \bar{\chi}(x),$$

where $\bar{\eta}$ and $\bar{\chi}$ are the quadratic and canonical additive characters of \mathbb{F}_p , respectively.

The lemmas presented below will turn out to be of use in the sequel.

Lemma 1 (See Theorems 5.15 [19]) *With the symbols and notation above, we have*

$$G(\eta, \chi) = (-1)^{m-1} (\sqrt{-1})^{\frac{(p-1)^2}{4}m} p^{\frac{m}{2}},$$

and

$$G(\bar{\eta}, \bar{\chi}) = (\sqrt{-1})^{\frac{(p-1)^2}{4}} p^{\frac{1}{2}}.$$

Lemma 2 (See Theorem 5.33 of [19]) *With the symbols and notation above. Let $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_{p^m}[x]$ with $a_2 \neq 0$. Then*

$$\sum_{x \in \mathbb{F}_{p^m}} \chi(f(x)) = \chi(a_0 - a_1^2(4a_2)^{-1}) \eta(a_2) G(\eta, \chi).$$

Let $d = \gcd(m, l)$ denote the greatest common divisor of m and l . Take $s = m/d$. In the sequel we will require the following lemma whose proof can be found in [4, 13, 24].

Lemma 3 *Let $S(a) = \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}(ax^{p^l+1})}$ and $d = \gcd(m, l)$. Let $v_2(\cdot)$ denote the 2-adic order function. Then $Q(x) = \text{Tr}(ax^{p^l+1})$ is a quadratic form and for any $a \in \mathbb{F}_{p^m}^*$,*

① *If $v_2(m) \leq v_2(k)$, then $\text{rank}(Q(x)) = m$ and*

$$S(a) = \begin{cases} \sqrt{(-1)^{\frac{p^d-1}{2}}} p^{\frac{m}{2}}, & \frac{p^m-1}{2} \text{ times,} \\ -\sqrt{(-1)^{\frac{p^d-1}{2}}} p^{\frac{m}{2}}, & \frac{p^m-1}{2} \text{ times.} \end{cases} \quad (1)$$

② *If $v_2(m) = v_2(k) + 1$, then $\text{rank}(Q(x)) = m$ or $m - 2d$ and*

$$S(a) = \begin{cases} -p^{\frac{m}{2}}, & \frac{p^d(p^m-1)}{p^d+1} \text{ times,} \\ p^{\frac{m}{2}+d}, & \frac{p^m-1}{p^d+1} \text{ times.} \end{cases} \quad (2)$$

③ *If $v_2(m) > v_2(k) + 1$, then $\text{rank}(Q(x)) = m$ or $m - 2d$ and*

$$S(a) = \begin{cases} p^{\frac{m}{2}}, & \frac{p^d(p^m-1)}{p^d+1} \text{ times,} \\ -p^{\frac{m}{2}+d}, & \frac{p^m-1}{p^d+1} \text{ times.} \end{cases} \quad (3)$$

The following lemma gives the value distribution of the exponential sum

$$T(a, b) = \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}_1^m(ax^{p^l+1}+bx^2)}.$$

Lemma 4 (See Lemma 2 and Theorem 1 of [20]) When (a, b) runs through $\mathbb{F}_{p^m}^2 \setminus \{(0, 0)\}$, the quadratic form $\text{Tr}_1^m(ax^{p^l+1} + bx^2)$ has possible rank m , $m-d$ or $m-2d$, and

(i) For s being odd, the exponential sum $T(a, b)$ has the following value distribution:

$$\begin{cases} \sqrt{(-1)^{\frac{p^d-1}{2}} p^{\frac{m}{2}}}, & |R_1| \text{ times}, \\ -\sqrt{(-1)^{\frac{p^d-1}{2}} p^{\frac{m}{2}}}, & |R_1| \text{ times}, \\ p^{\frac{m+d}{2}}, & |R_2| \text{ times}, \\ -p^{\frac{m+d}{2}}, & |R_3| \text{ times}, \\ \sqrt{(-1)^{\frac{p^d-1}{2}} p^{\frac{m+2d}{2}}}, & |R_4| \text{ times}, \\ -\sqrt{(-1)^{\frac{p^d-1}{2}} p^{\frac{m+2d}{2}}}, & |R_4| \text{ times}, \end{cases}$$

where $|R_i|$ is given by

$$\begin{cases} |R_1| = \frac{(p^{m+2d} - p^{m+d} - p^m + p^{2d})(p^m - 1)}{2(p^{2d} - 1)}, \\ |R_2| = \frac{1}{2}(p^{m-d} + p^{\frac{m-d}{2}})(p^m - 1), \\ |R_3| = \frac{1}{2}(p^{m-d} - p^{\frac{m-d}{2}})(p^m - 1), \\ |R_4| = \frac{(p^{m-d} - 1)(p^m - 1)}{2(p^{2d} - 1)}. \end{cases} \quad (4)$$

(ii) For s being even, the exponential sum $T(a, b)$ has the following value distribution:

$$\begin{cases} p^{\frac{m}{2}}, & |K_1| \text{ times}, \\ -p^{\frac{m}{2}}, & |K_2| \text{ times}, \\ \sqrt{(-1)^{\frac{p^d-1}{2}} p^{\frac{m+d}{2}}}, & |K_3| \text{ times}, \\ -\sqrt{(-1)^{\frac{p^d-1}{2}} p^{\frac{m+d}{2}}}, & |K_3| \text{ times}, \\ p^{\frac{m}{2}+d}, & |K_4| \text{ times}, \\ -p^{\frac{m}{2}+d}, & |K_5| \text{ times}. \end{cases}$$

where $|K_i|$ is given by

$$\begin{cases} |K_1| = \frac{(p^{m+2d} - p^{m+d} - p^m + p^{\frac{m}{2}+2d} - p^{\frac{m}{2}+d} + p^{2d})(p^m - 1)}{2(p^{2d} - 1)}, \\ |K_2| = \frac{(p^{m+2d} - p^{m+d} - p^m - p^{\frac{m}{2}+2d} + p^{\frac{m}{2}+d} + p^{2d})(p^m - 1)}{2(p^{2d} - 1)}, \\ |K_3| = \frac{1}{2}p^{m-d}(p^m - 1), \\ |K_4| = \frac{(p^{\frac{m}{2}} - 1)(p^{\frac{m}{2}-d} + 1)(p^m - 1)}{2(p^{2d} - 1)}, \\ |K_5| = \frac{(p^{\frac{m}{2}} + 1)(p^{\frac{m}{2}-d} - 1)(p^m - 1)}{2(p^{2d} - 1)}. \end{cases} \quad (5)$$

3 Main results

This section investigates the complete weight enumerators of cyclic codes by utilizing the value distributions of the corresponding exponential sums. A general strategy is given and then used to special codes C_1 , C_1 and C_2 , respectively, as depicted in Section 1.

3.1 The General Strategy

We set up our strategy for the general situation which will be used throughout this paper.

Let

$$f_{a_{00}, \dots, a_{kk}}(x) = \sum_{i,j=0}^k a_{ij} x^{p^i + p^j}$$

be a polynomial over \mathbb{F}_{p^m} , where $k \leq m-1$. It can be verified that $\text{Tr}_1^m(f_{a_{00}, \dots, a_{kk}}(x))$ is a quadratic form over \mathbb{F}_{p^m} . The rank of $\text{Tr}_1^m(f_{a_{00}, \dots, a_{kk}}(x))$ is denoted by r .

Consider the exponential sum

$$S(a_{00}, \dots, a_{kk}) = \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}_1^m(f_{a_{00}, \dots, a_{kk}}(x))}.$$

We suppose that the sum $S(a_{00}, \dots, a_{kk})$ has been completely determined by the quadratic form $\text{Tr}_1^m(f_{a_{00}, \dots, a_{kk}}(x))$ over \mathbb{F}_{p^m} . In addition, let $f_{r,\beta}$ denote the frequency of $S(a_{00}, \dots, a_{kk})$ taking the value $S_{r,\beta}$ with rank r , for $\beta \in J$, where J is an index set.

Now we focus on the complete weight enumerator of the code

$$C = \{(\text{Tr}_1^m(f_{a_{00}, \dots, a_{kk}}(x)))_{x \in \mathbb{F}_{p^m}^*} : a_{00}, \dots, a_{kk} \in \mathbb{F}_{p^m}\}. \quad (6)$$

If $a_{00} = \dots = a_{kk} = 0$, the corresponding codeword is the zero codeword, and the contribution to the complete weight enumerator is

$$w_0^{p^m-1}.$$

Now consider the case that some a_{ij} is nonzero for $1 \leq i, j \leq k$. Let $n_{a_{00}, \dots, a_{kk}}(\rho)$ denote the number of solutions $x \in \mathbb{F}_{p^m}^*$ such that $\text{Tr}_1^m(f_{a_{00}, \dots, a_{kk}}(x)) = \rho$, where $\rho \in \mathbb{F}_p$, i.e.,

$$n_{a_{00}, \dots, a_{kk}}(\rho) = \#\{x \in \mathbb{F}_{p^m}^* : \text{Tr}_1^m(f_{a_{00}, \dots, a_{kk}}(x)) = \rho\}.$$

Then, the contributions of such terms to the complete weight enumerator are of the form

$$\prod_{\rho=0}^{p-1} w_\rho^{n_{a_{00}, \dots, a_{kk}}(\rho)},$$

and we only need to compute the frequency of each such term and the value of $n_{a_{00}, \dots, a_{kk}}(\rho)$ which will yield the complete weight enumerator of the code.

Consider the number of solutions $x \in \mathbb{F}_{p^m}$ such that $\text{Tr}_1^m(f_{a_{00}, \dots, a_{kk}}(x)) = \rho$, which is given by

$$N_{a_{00}, \dots, a_{kk}}(\rho) = \#\{x \in \mathbb{F}_{p^m} : \text{Tr}_1^m(f_{a_{00}, \dots, a_{kk}}(x)) = \rho\}.$$

It is straightforward that

$$n_{a_{00}, \dots, a_{kk}}(\rho) = \begin{cases} N_{a_{00}, \dots, a_{kk}}(\rho) - 1, & \text{if } \rho = 0, \\ N_{a_{00}, \dots, a_{kk}}(\rho), & \text{otherwise.} \end{cases} \quad (7)$$

Therefore, it suffices to study the value of $N_{a_{00}, \dots, a_{kk}}(\rho)$, which is determined by

$$\begin{aligned} N_{a_{00}, \dots, a_{kk}}(\rho) &= \frac{1}{p} \sum_{x \in \mathbb{F}_{p^m}} \sum_{y \in \mathbb{F}_p} \zeta_p^{y(\text{Tr}_1^m(f_{a_{00}, \dots, a_{kk}}(x)) - \rho)} \\ &= p^{m-1} + \frac{1}{p} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{y\rho} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{y\text{Tr}_1^m(f_{a_{00}, \dots, a_{kk}}(x))} \\ &= p^{m-1} + \frac{1}{p} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{y\rho} S(ya_{00}, \dots, ya_{kk}) \\ &= p^{m-1} + \frac{1}{p} S(a_{00}, \dots, a_{kk}) \sum_{y \in \mathbb{F}_p^*} \zeta_p^{y\rho} \bar{\eta}(y^r), \end{aligned} \quad (8)$$

where the last equal sign holds since

$$S(ya_{00}, \dots, ya_{kk}) = \bar{\eta}(y^r) S(a_{00}, \dots, a_{kk})$$

and $\bar{\eta}$ is the quadratic character over \mathbb{F}_p .

If $\rho = 0$, Equation (8) shows that

$$\begin{aligned} N_{a_{00}, \dots, a_{kk}}(0) &= p^{m-1} + \frac{1}{p} S(a_{00}, \dots, a_{kk}) \sum_{y \in \mathbb{F}_p^*} \bar{\eta}(y^r) \\ &= \begin{cases} p^{m-1} + \frac{p-1}{p} S(a_{00}, \dots, a_{kk}), & \text{if } r \text{ even,} \\ p^{m-1}, & \text{if } r \text{ odd.} \end{cases} \end{aligned} \quad (9)$$

and consequently

$$n_{a_{00}, \dots, a_{kk}}(0) = \begin{cases} p^{m-1} + \frac{p-1}{p} S(a_{00}, \dots, a_{kk}) - 1, & \text{if } r \text{ even,} \\ p^{m-1} - 1, & \text{if } r \text{ odd.} \end{cases} \quad (10)$$

If $\rho \in \mathbb{F}_p^*$, it follows from Equations (7) and (8) that

$$\begin{aligned} n_{a_{00}, \dots, a_{kk}}(\rho) &= N_{a_{00}, \dots, a_{kk}}(\rho) \\ &= \begin{cases} p^{m-1} - \frac{1}{p} S(a_{00}, \dots, a_{kk}), & \text{if } r \text{ even,} \\ p^{m-1} + \frac{1}{p} \bar{\eta}(\rho) S(a_{00}, \dots, a_{kk}) G(\bar{\eta}, \bar{\chi}), & \text{if } r \text{ odd.} \end{cases} \end{aligned} \quad (11)$$

By assumption that $f_{r,\beta}$ to be the frequency of $S(a_{00}, \dots, a_{kk})$ taking the value $S_{r,\beta}$ with rank r , each term $\prod_{\rho=0}^{p-1} w_\rho^{n_{a_{00}, \dots, a_{kk}}(\rho)}$ appears $f_{r,\beta}$ times according to the value of $S(a_{00}, \dots, a_{kk})$ with rank r . Clearly, $n_{a_{00}, \dots, a_{kk}}(\rho)$ is related to $S_{r,\beta}$ and thus we denote it by $n_{a_{00}, \dots, a_{kk}}(\rho; S_{r,\beta})$ to show this. Therefore, the complete weight enumerator for the code C is

$$\text{CWE}(C) = w_0^{p^m-1} + \sum_{r,\beta} f_{r,\beta} \prod_{\rho=0}^{p-1} w_\rho^{n_{a_{00}, \dots, a_{kk}}(\rho; S_{r,\beta})}.$$

3.2 The complete weight enumerator of the code C_1

Recall that

$$C_1 = \{c_2(a) = (\text{Tr}_1^m(ax^{p^l+1}))_{x \in \mathbb{F}_{p^m}} : a \in \mathbb{F}_{p^m}\},$$

which is a special case of (6).

Now we deal with the complete weight enumerator of the code C_1 by using the exponential sum

$$S(a) = \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}_1^m(ax^{p^l+1})}.$$

Theorem 1 *With notation given before.*

(i) *Assume that $m \neq 2l$. Then C_1 is a $[p^m - 1, m]$ cyclic code over \mathbb{F}_p and its complete weight enumerator is shown as follows:*

① *If $0 = v_2(m) \leq v_2(l)$, then*

$$\begin{aligned} \text{CWE}(C_1) &= w_0^{p^m-1} + \frac{p^m - 1}{2} w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} + \bar{\eta}(\rho)p^{\frac{m-1}{2}}} \\ &\quad + \frac{p^m - 1}{2} w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} - \bar{\eta}(\rho)p^{\frac{m-1}{2}}}. \end{aligned} \quad (12)$$

② *If $1 \leq v_2(m) \leq v_2(l)$, then*

$$\begin{aligned} \text{CWE}(C_1) &= w_0^{p^m-1} + \frac{p^m - 1}{2} w_0^{p^{m-1}-1+(p-1)p^{\frac{m-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p^{\frac{m-2}{2}}} \\ &\quad + \frac{p^m - 1}{2} w_0^{p^{m-1}-1-(p-1)p^{\frac{m-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p^{\frac{m-2}{2}}}. \end{aligned} \quad (13)$$

③ *If $v_2(m) = v_2(l) + 1$, then*

$$\text{CWE}(C_1) = w_0^{p^m-1} + \frac{p^d(p^m - 1)}{p^d + 1} w_0^{p^{m-1}-1-(p-1)p^{\frac{m-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p^{\frac{m-2}{2}}}$$

$$+ \frac{p^m - 1}{p^d + 1} w_0^{p^{m-1} - 1 + (p-1)p^{\frac{m+2d-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} - p^{\frac{m+2d-2}{2}}}. \quad (14)$$

④ If $v_2(m) > v_2(l) + 1$, then

$$\begin{aligned} \text{CWE}(C_1) &= w_0^{p^m - 1} + \frac{p^d(p^m - 1)}{p^d + 1} w_0^{p^{m-1} - 1 + (p-1)p^{\frac{m-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} - p^{\frac{m-2}{2}}} \\ &\quad + \frac{p^m - 1}{p^d + 1} w_0^{p^{m-1} - 1 - (p-1)p^{\frac{m+2d-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} + p^{\frac{m+2d-2}{2}}}. \end{aligned} \quad (15)$$

(ii) Assume that $m = 2l$. Then C_1 is a $[p^m - 1, m/2]$ cyclic code over \mathbb{F}_p and its complete weight enumerator is given by

$$\text{CWE}(C_1) = w_0^{p^m - 1} + (p^{\frac{m}{2}} - 1) w_0^{p^{m-1} - 1 - (p-1)p^{\frac{m-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} + p^{\frac{m-2}{2}}}. \quad (16)$$

Proof (i) Assume that $m \neq 2l$. We only give the proof for the case $0 = v_2(m) \leq v_2(l)$ since other cases are similar.

Clearly $a = 0$ gives the zero codeword and the contribution to the complete weight enumerator is

$$w_0^{p^m - 1}.$$

Consider $a \in \mathbb{F}_{p^m}^*$. Let

$$N_a(\rho) = \#\{x \in \mathbb{F}_{p^m} : \text{Tr}_1^m(ax^{p^l+1}) = \rho\}.$$

and

$$n_a(\rho) = \#\{x \in \mathbb{F}_{p^m}^* : \text{Tr}_1^m(ax^{p^l+1}) = \rho\}.$$

Note that $r = m$ and m is odd. By Equations (9) and (11), we have

$$N_a(0) = p^{m-1},$$

and for a fixed $\rho \in \mathbb{F}_p^*$,

$$N_a(\rho) = p^{m-1} + \frac{1}{p} \bar{\eta}(\rho) S(a) G(\bar{\eta}, \bar{\chi}).$$

It then follows from Equation (1) and Lemma 1 that

$$N_a(\rho) = \begin{cases} p^{m-1} + \bar{\eta}(\rho) \sqrt{-1}^{\frac{(p-1)^2}{4} + \frac{p^d-1}{2}} p^{\frac{m-1}{2}}, & \frac{p^m-1}{2} \text{ times,} \\ p^{m-1} - \bar{\eta}(\rho) \sqrt{-1}^{\frac{(p-1)^2}{4} + \frac{p^d-1}{2}} p^{\frac{m-1}{2}}, & \frac{p^m-1}{2} \text{ times.} \end{cases}$$

Note that $\frac{(p-1)^2}{4} + \frac{p^d-1}{2}$ is an even integer. This implies that

$$N_a(\rho) = \begin{cases} p^{m-1} + \bar{\eta}(\rho) p^{\frac{m-1}{2}}, & \frac{p^m-1}{2} \text{ times,} \\ p^{m-1} - \bar{\eta}(\rho) p^{\frac{m-1}{2}}, & \frac{p^m-1}{2} \text{ times.} \end{cases}$$

By Equation (7) and the above analysis, the result given by Equation (12) holds for the case $0 = v_2(m) \leq v_2(l)$.

(ii) Assume that $m = 2l$.

Let

$$K = \{x \in \mathbb{F}_{p^m} \mid x^{p^l} + x = 0\}.$$

Note that $c_2(a) = c_2(a + \tau)$ for any $\tau \in K$ and $c_2(a) \in C_1$. Hence, C_1 is degenerate with dimension $m/2$ over \mathbb{F}_p .

Clearly $|K| = p^{\frac{m}{2}}$ and $v_2(m) = v_2(l) + 1$. Substituting $d = m/2$ to Equation (14) and dividing each A_i by $p^{\frac{m}{2}}$, we get the result given by (16).

This finishes the proof of Theorem 1. \square

Example 1 (i) Let $m = 3$, $k = 1$, $p = 5$. This corresponds to the case $0 = v_2(m) \leq v_2(l)$. Magma works out that the complete weight enumerator for the code C_1 is

$$w_0^{124} + 62w_0^{24}w_1^{30}w_2^{20}w_3^{20}w_4^{30} + 62w_0^{24}w_1^{20}w_2^{30}w_3^{30}w_4^{20}.$$

(ii) Let $m = 6$, $k = 2$, $p = 3$. This corresponds to the case $1 \leq v_2(m) \leq v_2(l)$. Magma shows that the complete weight enumerator for the code C_1 is

$$w_0^{728} + 364w_0^{260}w_1^{234}w_2^{234} + 364w_0^{224}w_1^{252}w_2^{252}.$$

(iii) Let $m = 6$, $k = 1$, $p = 3$. This corresponds to the case $v_2(m) = v_2(l) + 1$ and $m \neq 2l$. Magma computes that the complete weight enumerator for the code C_1 is

$$w_0^{728} + 182w_0^{296}w_1^{216}w_2^{216} + 546w_0^{224}w_1^{252}w_2^{252}.$$

(iv) Let $m = 4$, $k = 1$, $p = 3$. This corresponds to the case $v_2(m) > v_2(l) + 1$. With the help of Magma, we know that the complete weight enumerator for the code C_1 is

$$w_0^{80} + 60w_0^{32}w_1^{24}w_2^{24} + 20w_0^8w_1^{36}w_2^{36}.$$

(v) Let $m = 2$, $k = 1$, $p = 3$. This corresponds to the case $m = 2l$. Magma works out that the complete weight enumerator for the code C_1 is

$$w_0^8 + 2w_1^4w_2^4.$$

These experimental results coincide with the complete weight enumerators in Theorem 1.

3.3 The complete weight enumerator for the code C_2

Recall that

$$C_2 = \{c_3(a, b) = (\text{Tr}_1^m(ax^{p^l+1} + bx^2))_{x \in \mathbb{F}_{p^m}^*} : a, b \in \mathbb{F}_{p^m}\}.$$

Now we present the complete weight enumerator of the code C_2 by employing the exponential sum

$$T(a, b) = \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}_1^m(ax^{p^l+1} + bx^2)}.$$

Theorem 2 *With notation given before. Let $|R_i|$ and $|K_i|$ be given by (4) and (5), respectively.*

(i) *Assume that $m \neq 2l$. Then C_2 is a $[p^m - 1, 2m]$ cyclic code over \mathbb{F}_p and its complete weight enumerator is shown as follows:*

① *For the case of s and d both being odd, we have*

$$\begin{aligned} \text{CWE}(C_2) &= w_0^{p^m-1} + |R_1|w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} + \bar{\eta}(\rho)p^{\frac{m-1}{2}}} \\ &\quad + |R_1|w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} - \bar{\eta}(\rho)p^{\frac{m-1}{2}}} \\ &\quad + |R_2|w_0^{p^{m-1}-1+(p-1)p^{\frac{m+d-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p^{\frac{m+d-2}{2}}} \\ &\quad + |R_3|w_0^{p^{m-1}-1-(p-1)p^{\frac{m+d-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p^{\frac{m+d-2}{2}}} \\ &\quad + |R_4|w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} + \bar{\eta}(\rho)p^{\frac{m+2d-1}{2}}} \\ &\quad + |R_4|w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} - \bar{\eta}(\rho)p^{\frac{m+2d-1}{2}}} \end{aligned}$$

② *For the case of s being odd and d being even, we have*

$$\begin{aligned} \text{CWE}(C_2) &= w_0^{p^m-1} + |R_1|w_0^{p^{m-1}-1+(p-1)p^{\frac{m-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p^{\frac{m-2}{2}}} \\ &\quad + |R_1|w_0^{p^{m-1}-1-(p-1)p^{\frac{m-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p^{\frac{m-2}{2}}} \\ &\quad + |R_2|w_0^{p^{m-1}-1+(p-1)p^{\frac{m+d-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p^{\frac{m+d-2}{2}}} \end{aligned}$$

$$\begin{aligned}
& + |R_3| w_0^{p^{m-1}-1-(p-1)p \frac{m+d-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p \frac{m+d-2}{2}} \\
& + |R_4| w_0^{p^{m-1}-1+(p-1)p \frac{m+2d-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p \frac{m+2d-2}{2}} \\
& + |R_4| w_0^{p^{m-1}-1-(p-1)p \frac{m+2d-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p \frac{m+2d-2}{2}}
\end{aligned}$$

③ For the case of s being even and d being odd, we have

$$\begin{aligned}
\text{CWE}(C_2) &= w_0^{p^{m-1}} + |K_1| w_0^{p^{m-1}-1+(p-1)p \frac{m-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p \frac{m-2}{2}} \\
& + |K_2| w_0^{p^{m-1}-1-(p-1)p \frac{m-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p \frac{m-2}{2}} \\
& + |K_3| w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+\bar{\eta}(\rho)p \frac{m+d-1}{2}} \\
& + |K_3| w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-\bar{\eta}(\rho)p \frac{m+d-1}{2}} \\
& + |K_4| w_0^{p^{m-1}-1+(p-1)p \frac{m+2d-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p \frac{m+2d-2}{2}} \\
& + |K_5| w_0^{p^{m-1}-1-(p-1)p \frac{m+2d-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p \frac{m+2d-2}{2}}
\end{aligned}$$

④ For the case of s and d both being even, we have

$$\begin{aligned}
\text{CWE}(C_2) &= w_0^{p^{m-1}} + |K_1| w_0^{p^{m-1}-1+(p-1)p \frac{m-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p \frac{m-2}{2}} \\
& + |K_2| w_0^{p^{m-1}-1-(p-1)p \frac{m-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p \frac{m-2}{2}} \\
& + |K_3| w_0^{p^{m-1}-1+(p-1)p \frac{m+d-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p \frac{m+d-2}{2}} \\
& + |K_3| w_0^{p^{m-1}-1-(p-1)p \frac{m+d-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p \frac{m+d-2}{2}} \\
& + |K_4| w_0^{p^{m-1}-1+(p-1)p \frac{m+2d-2}{2}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p \frac{m+2d-2}{2}}
\end{aligned}$$

$$+|K_5|w_0^{p^{m-1}-1-(p-1)p^{\frac{m+2d-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p^{\frac{m+2d-2}{2}}}$$

(ii) Assume that $m = 2l$. Then C_2 is a $[p^m - 1, 3m/2]$ cyclic code over \mathbb{F}_p and

① For the case of d being odd, we have

$$\begin{aligned} \text{CWE}(C_2) &= w_0^{p^{m-1}} + \frac{1}{2}p^{\frac{m}{2}}(p^m - 1)w_0^{p^{m-1}+(p-1)p^{\frac{m-2}{2}}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p^{\frac{m-2}{2}}} \\ &\quad + \frac{1}{2}p^{\frac{m}{2}}(p^{\frac{m}{2}} - 1)^2 w_0^{p^{m-1}-1-(p-1)p^{\frac{m-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p^{\frac{m-2}{2}}} \\ &\quad + \frac{1}{2}(p^m - 1)w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+\bar{\eta}(\rho)p^{\frac{3m-2}{4}}} \\ &\quad + \frac{1}{2}(p^m - 1)w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-\bar{\eta}(\rho)p^{\frac{3m-2}{4}}} \end{aligned}$$

② For the case of d being even, we have

$$\begin{aligned} \text{CWE}(C_2) &= w_0^{p^{m-1}} + \frac{1}{2}p^{\frac{m}{2}}(p^m - 1)w_0^{p^{m-1}-1+(p-1)p^{\frac{m-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p^{\frac{m-2}{2}}} \\ &\quad + \frac{1}{2}p^{\frac{m}{2}}(p^{\frac{m}{2}} - 1)^2 w_0^{p^{m-1}-1-(p-1)p^{\frac{m-2}{2}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p^{\frac{m-2}{2}}} \\ &\quad + \frac{1}{2}(p^m - 1)w_0^{p^{m-1}-1+(p-1)p^{\frac{3m-4}{4}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}-p^{\frac{3m-4}{4}}} \\ &\quad + \frac{1}{2}(p^m - 1)w_0^{p^{m-1}-1-(p-1)p^{\frac{3m-4}{4}}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1}+p^{\frac{3m-4}{4}}} \end{aligned}$$

Proof (i) Assume that $m \neq 2l$. We only give the proof for the case of s and d both being odd, since other cases are similar to get.

Clearly $a = b = 0$ gives the zero codeword and the contribution to the complete weight enumerator is

$$w_0^{p^m-1}.$$

Consider a or b is nonzero. Let

$$n_{a,b}(\rho) = \#\{x \in \mathbb{F}_{p^m}^* : \text{Tr}(ax^{p^l+1} + bx^2) = \rho\},$$

and

$$N_{a,b}(\rho) = \#\{x \in \mathbb{F}_{p^m} : \text{Tr}(ax^{p^l+1} + bx^2) = \rho\},$$

where $\rho \in \mathbb{F}_p$.

Recall that $d = \gcd(m, l)$ and $s = m/d$. Since both m and d are odd, we have m and $m - 2d$ are odd and $m - d$ is even.

The value of $N_{a,b}(\rho)$ will be calculated by distinguishing the following subcases.

Case 1: $r = m$.

In this case, Lemma 4 shows that

$$T(a, b) = \begin{cases} \sqrt{(-1)^{\frac{p^d-1}{2}} p^{\frac{m}{2}}}, & |R_1| \text{ times}, \\ -\sqrt{(-1)^{\frac{p^d-1}{2}} p^{\frac{m}{2}}}, & |R_1| \text{ times}. \end{cases}$$

It follows from Equation (8) that

$$\begin{aligned} N_{a,b}(\rho) &= p^{m-1} + \frac{1}{p} T(a, b) \sum_{y \in \mathbb{F}_p^*} \zeta_p^{y\rho} \bar{\eta}(y^r), \\ &= p^{m-1} + \frac{1}{p} T(a, b) \sum_{y \in \mathbb{F}_p^*} \zeta_p^{y\rho} \bar{\eta}(y), \\ &= \begin{cases} p^{m-1}, & \text{if } \rho = 0, \\ p^{m-1} + \frac{1}{p} \bar{\eta}(\rho) T(a, b) G(\bar{\eta}, \bar{\chi}), & \text{otherwise.} \end{cases} \end{aligned}$$

Then

$$n_{a,b}(\rho) = \begin{cases} p^{m-1} - 1, & \text{if } \rho = 0, \\ p^{m-1} + \frac{1}{p} \bar{\eta}(\rho) T(a, b) G(\bar{\eta}, \bar{\chi}), & \text{otherwise.} \end{cases}$$

By Lemma 1, the contributions of such terms to the complete weight enumerator is then

$$|R_1| w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} + \bar{\eta}(\rho) p^{\frac{m-1}{2}}} + |R_1| w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} - \bar{\eta}(\rho) p^{\frac{m-1}{2}}}.$$

Case 2: $r = m - d$.

In this case, we have

$$T(a, b) = \begin{cases} p^{\frac{m+d}{2}}, & |R_2| \text{ times}, \\ -p^{\frac{m+d}{2}}, & |R_3| \text{ times}. \end{cases}$$

Equation (8) yields that

$$\begin{aligned} N_{a,b}(\rho) &= p^{m-1} + \frac{1}{p} T(a, b) \sum_{y \in \mathbb{F}_p^*} \zeta_p^{y\rho} \\ &= \begin{cases} p^{m-1} + \frac{p-1}{p} T(a, b), & \text{if } \rho = 0, \\ p^{m-1} - \frac{1}{p} T(a, b), & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore

$$n_{a,b}(\rho) = \begin{cases} p^{m-1} - 1 + \frac{p-1}{p}T(a,b), & \text{if } \rho = 0, \\ p^{m-1} - \frac{1}{p}T(a,b), & \text{otherwise.} \end{cases}$$

The contributions of such terms to the complete weight enumerator is

$$\begin{aligned} & |R_2| w_0^{p^{m-1} + (p-1)p^{\frac{m+d-2}{2}-1}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} - p^{\frac{m+d-2}{2}}} \\ & + |R_3| w_0^{p^{m-1} - (p-1)p^{\frac{m+d-2}{2}-1}} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} + p^{\frac{m+d-2}{2}}} \end{aligned}$$

Case 3: $r = m - 2d$.

In this case, we have

$$T(a,b) = \begin{cases} \sqrt{(-1)^{\frac{p^d-1}{2}} p^{\frac{m+2d}{2}}}, & |R_4| \text{ times,} \\ -\sqrt{(-1)^{\frac{p^d-1}{2}} p^{\frac{m+2d}{2}}}, & |R_4| \text{ times.} \end{cases}$$

By a similar discussion to *Case 1*, we have

$$n_{a,b}(\rho) = \begin{cases} p^{m-1} - 1, & \text{if } \rho = 0, \\ p^{m-1} + \frac{1}{p}\bar{\eta}(\rho)T(a,b)G(\bar{\eta}, \bar{\chi}), & \text{otherwise.} \end{cases}$$

Therefore, the contributions of such terms to the complete weight enumerator is then

$$|R_4| w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} + \bar{\eta}(\rho)p^{\frac{m+2d-1}{2}}} + |R_4| w_0^{p^{m-1}-1} \prod_{\rho \in \mathbb{F}_p^*} w_\rho^{p^{m-1} - \bar{\eta}(\rho)p^{\frac{m+2d-1}{2}}}.$$

The desired conclusion then follows immediately from the above arguments.

(ii) Assume that $m = 2l$.

Let

$$K = \{x \in \mathbb{F}_{p^m} \mid x^{p^l} + x = 0\}.$$

Note that $c_3(a,b) = c_3(a+\tau,b)$ for any $\tau \in K$ and $c_3(a,b) \in C_2$. Hence, C_2 is degenerate with dimension $3m/2$ over \mathbb{F}_p .

Clearly $|K| = p^{\frac{m}{2}}$ and $s = 2$. Substituting $d = m/2$ to the case s being even and dividing each frequency by $p^{\frac{m}{2}}$, we get the desired results.

This finishes the proof of Theorem 2. \square

Example 2 (i) Let $m = 3$, $k = 1$, $p = 3$. Then $s = 3$ and $d = 1$. Magma works out that C_2 is a $[26, 6, 12]$ cyclic code with the complete weight enumerator

$$w_0^{26} + 156w_0^{14}w_1^6w_2^6 + 13w_0^8w_1^{18} + 234w_0^8w_1^{12}w_2^6 \\ + 234w_0^8w_1^6w_2^{12} + 13w_0^8w_2^{18} + 78w_0^2w_1^{12}w_2^{12}.$$

(ii) Let $m = 6$, $k = 2$, $p = 3$. Then $s = 3$ and $d = 2$. Magma works out that C_2 is a $[728, 12, 324]$ cyclic code with the complete weight enumerator

$$w_0^{728} + 364w_0^{404}w_1^{162}w_2^{162} + 32760w_0^{296}w_1^{216}w_2^{216} + 235872w_0^{260}w_1^{234}w_2^{234} \\ + 235872w_0^{224}w_1^{252}w_2^{252} + 26208w_0^{188}w_1^{270}w_2^{270} + 364w_0^{80}w_1^{324}w_2^{324}.$$

(iii) Let $m = 4$, $k = 2$, $p = 3$. Then $s = 2$ and $d = 2$. Magma works out that C_2 is a $[80, 6, 36]$ cyclic code with the complete weight enumerator

$$w_0^{80} + 40w_0^{44}w_1^{18}w_2^{18} + 360w_0^{32}w_1^{24}w_2^{24} + 288w_0^{20}w_1^{30}w_2^{30} + 40w_0^8w_1^{36}w_2^{36}.$$

These experimental results coincide with the complete weight enumerators in Theorem 2.

4 Conclusion and remarks

In this paper, we concentrated on the complete weight enumerators of cyclic codes. A general strategy was proposed by using exponential sums and then the complete weight enumerators of three classes of cyclic codes were explicitly determined. In addition, one can get the weight distributions of the codes through their complete weight enumerators.

It should be noted that the exponential sums are known in a few cases. Hence the complete weight enumerator of most cyclic codes cannot be explicitly presented. We mention that the complete weight enumerators are still open for most cyclic codes and it will be a good research problem to construct more cyclic codes and determine their complete weight enumerators and weight distributions as well. We leave this for future work.

Acknowledgements The work of Zheng-An Yao is partially supported by the NSFC (Grant No.11271381), the NSFC (Grant No.11431015) and China 973 Program (Grant No. 2011CB808000). This work is also partially supported by the NSFC (Grant No. 61472457) and Guangdong Natural Science Foundation (Grant No. 2014A030313161).

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